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Short Communication

A study of nonlinear oscillators with $u^{1/3}$ force by He's variational iteration method

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Abstract

This paper applies He's variational iteration method to determine the periodic solutions of oscillators in a $u^{1/3}$ force. With the procedure, the excellent approximate frequencies and the corresponding periodic solutions can easily be obtained.

1. Introduction

In nonlinear analysis, perturbation methods are well-established tools to study diverse aspects of nonlinear problems. Surveys of the early literature with numerous references, and useful bibliographies, have been given by Nayfeh [1], Mickens [2], Jordan and Smith [3] and Hagedorn [4]. However, the use of perturbation theory in many important practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings. Such a new technique should work over a large range of parameters and yield accurate analytical approximate solutions beyond the coverage and ability of the classical perturbation methods.

For example, variational methods have been, and continue to be, accepted tools for nonlinear oscillators, for example, D'Acunto applied He's variational method [5,6] to various nonlinear oscillators. He [7] himself applied a new variational method to a kind of general nonlinear oscillators. He's variational iteration method [8], is successfully and easily used to solve some class of nonlinear problems. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. Relatively comprehensive survey on the method and its applications can be found in Refs. [9–18], monograph [19] and the references therein.

There also exists a wide range of literature dealing with the approximate determination of periodic solutions for nonlinear problems by using a mixture of methodologies [20–32].

The purpose of this paper is the determination of the periodic solutions to nonlinear oscillator equations for which the elastic restoring forces are non-polynomial functions of the displacement by applying He's

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variational iteration method. This class of equations represents a new class of nonlinear oscillating systems, which were first studied in detail by Mickens [33].

2. He's variational iteration method

Currently, we will study the properties of the periodic solutions to certain nonlinear oscillators by applying He's variational iteration method for which the elastic restoring forces are non-polynomial functions of the displacement. In particular, this term is chosen to be

$$f(x) = -x^{1/3}. \tag{1}$$

As it is well known that the classical perturbation methods [1–4] give uniformly valid asymptotic expansions for the periodic solutions of weakly nonlinear oscillators, in general, the technique is not applicable in case of strongly nonlinear terms or elastic restoring forces are non-polynomial functions. Therefore, the work reported here applies He's variational iteration method which also works for strongly nonlinear systems as well as the nonlinear systems with the elastic restoring forces are non-polynomial functions of displacement.

Now, to illustrate the basic concept of He's variational iteration method, we consider the following general nonlinear differential equation given in the form:

$$Lu(t) + Nu(t) = g(t), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. We can construct a correction functional according to the variational method as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\mathrm{d}\xi, \tag{3}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, namely $\delta\tilde{u}_n = 0$ [8–10].

In the following examples, we will illustrate the usefulness and effectiveness of the proposed technique.

Example 1. Now, consider the following nonlinear oscillator which was first studied in detail by Mickens [33]:

$$u'' + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{4}$$

By applying harmonic balance method and using the first-order approximate solution

$$u_0 \simeq A \cos \omega t \tag{5}$$

to Eq. (1), Mickens determined angular frequency, ω, as

$$\omega = \left(\frac{4}{3A^2}\right)^{1/6} \approx 1.04912A^{-1/3}. \tag{6}$$

Mickens [34] also used the second-order harmonic balance approximation to the periodic solution of Eq. (1) and determined ω as

$$\omega = \frac{1}{\left[\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right)\bar{z} + \left(\frac{243}{2}\right)\bar{z}^2\right]^{1/6}}\left(\frac{1+\bar{z}}{A}\right)^{1/3}, \tag{7}$$

where $\bar{z}$ is one of the solution having the smallest absolute magnitude of the polynomial equation

$$(1701)z^3 - (27)z^2 + (51)z + 1 = 0.$$

Comparing Eq. (3) with Eq. (4), it is clearly seen that the second harmonic balance approximation only provides small corrections to the periodic solution obtained in the first approximation and is negligible. This is the expected result.

More recently, He [19], and Xu [35], determined ω as in Eq. (3) by applying homotopy perturbation method and bookkeeping parameter method [19], respectively.

Now, to apply variational iteration method, we write correction functional of Eq. (4) as

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\,\omega(s-t)\left\{u_n''(s) + u_n^{1/3}(s)\right\}\mathrm{d}s. \tag{8}$$

We try the input of starting function as

$$u_0 = A\,\cos\,\omega t \tag{9}$$

and we have the first iteration as

$$u_1(t) = A\,\cos\,\omega t + \frac{1}{\omega}\int_0^t \sin\,\omega(s-t)\left\{-A\omega^2\,\cos\,\omega t + (A\,\cos\,\omega s)^{1/3}\right\}\mathrm{d}s. \tag{10}$$

Fourier series representation is needed for $(\cos\omega t)^{1/3}$. It has been calculated [36] and is given by

$$(\cos\,\omega t)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\omega t, \tag{11}$$

$$a_{2n+1} = \frac{3\Gamma(\frac{7}{3})}{2^{4/3}\Gamma(n+\frac{5}{3})\Gamma(\frac{2}{3}-n)},$$

with $a_1 = 1.159595266963929$. Therefore, the first several terms are

$$(\cos\,\omega t)^{1/3} = a_1\left[\cos\,\omega t - \frac{\cos\,3\omega t}{5} + \frac{\cos\,5\omega t}{10} - \frac{7\cos\,7\omega t}{110} + \frac{\cos\,9\omega t}{22} - \frac{13\cos\,11\omega t}{374} + \dots\right]. \tag{12}$$

Substituting Eq. (12) into Eq. (10) yields

$$u_1(t) = A\,\cos\,\omega t + \frac{1}{\omega}\int_0^t \sin\,\omega(s-t)\left\{-A\omega^2\,\cos\,\omega s + A^{1/3}a_1\left[\cos\,\omega s - \frac{\cos\,3\omega s}{5} + \cdots\right]\right\}\mathrm{d}s = 0. \tag{13}$$

The requirement of no secular term gives

$$-A\omega^2 + A^{1/3}a_1 = 0 \tag{14}$$

and the angular frequency determined as

$$\omega = \frac{1.0768}{A^{1/3}}. \tag{15}$$

We, therefore, obtain the following approximated period

$$T = \frac{2\pi A^{1/3}}{1.0768} = 5.835A^{1/3}. \tag{16}$$

For purpose of comparison, Mickens' first-order harmonic balance method [33], He's homotpy perturbation solution with first-order approximation [19] and Xu's solution [35] by utilizing bookkeeping method [19] give the frequency as $A^{1/3}\omega = 1.0491$. Mickens' second-order harmonic balance [34] gives the calculated value of the frequency as $A^{1/3}\omega = 1.0704$. Öziş and Yıldırım's modified Lindstedt–Poincaré method solution [37] agrees exactly with the present solution (15). The exact value [38] of the frequency read $A^{1/3}\omega_{\mathrm{ex}} = 1.070451$. Hence, the exact period is

$$T_{\mathrm{ex}} = \frac{2\pi A^{1/3}}{1.070451} = 5.86966A^{1/3}. \tag{17}$$

It can be easily shown that the maximal relative error is less than 0.59%.

Example 2. The second equation to be studied is a modified version of the van der Pol equation [39], i.e.,

$$u'' + u^{1/3} = \varepsilon(1-u^2)u', \quad u(0) = A, \quad u'(0) = 0. \tag{18}$$

Its correction functional can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\,\omega(s-t)\Big\{u_n(s) + u_n^{1/3}(s) - \varepsilon(1-u_n^2(s)u_n'(s)\Big\}\mathrm{d}s. \tag{19}$$

If we seek the input of starting function, $u_0 = A\cos\omega t$, we have

$$u_1(t) = A\,\cos\,\omega t + \frac{1}{\omega}\int_0^t \sin(s-t)\Big\{-A\omega^2\,\cos\,\omega s + (A\,\cos\,\omega s)^{1/3} - \varepsilon(1-(A\,\cos\,\omega s)^2)(-A\omega)\sin\,\omega s\Big\}\mathrm{d}s. \tag{20}$$

Replacing Fourier expansion of $(\cos\omega t)^{1/3}$ from Eq. (12), it follows that Eq. (20) becomes

$$\begin{aligned} u_1(t) = A\,\cos\,\omega t + \frac{1}{\omega}\int_0^t \sin(s-t)\Big\{&-A\omega^2\,\cos\,\omega s + A^{1/3}a_1\left[\cos\,\omega s - \frac{\cos\,\omega s}{5} + \cdots\right] \\ &+\varepsilon A\omega\left(1-\frac{A^2}{4}\right)\sin\,\omega s - \frac{\varepsilon A^3\omega}{4}\sin\,3\omega s\Big\}\mathrm{d}s = 0. \end{aligned} \tag{21}$$

The requirement of no secular term gives

$$1 - \frac{A^2}{4} = 0 \quad \text{and} \quad -A\omega^2 + A^{1/3}a_1 = 0 \tag{22}$$

and therefore, we obtain,

$$A = 2 \quad \text{and} \quad \omega = \frac{1.0768}{A^{1/3}} = \frac{1.0768}{2^{1/3}} = 0.8547. \tag{23}$$

We, therefore, obtain the following approximated period:

$$T = \frac{2\pi}{0.8547}, \tag{24}$$

which agrees exactly with Öziş and Yıldırım's [37] solution and Mickens' solution [39].

3. Conclusion

In summary, we have demonstrated the applicability of the method for solving nonlinear problems with fractional order with the help of some concrete examples. The method is extremely simple, easy to use and is very accurate for entire solution domain. Also, the method is a powerful tool to search for approximate solutions of various linear/nonlinear problems with integral/fractional order and/or strong nonlinearity. To our knowledge, the method can also be extended to wide range of problems such as (singular) nonlinear boundary value problems, delay differential equations, autonomous systems and other problems of mathematical physics. We think that the method have great potential which still needs further development.

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